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TESTING RELIABILITY: FACTORIAL DESIGN WITH DATA FROM A LOG-EPSILON-SKEW-NORMAL DISTRIBUTION

Abstract.*Experimental designs are among the most popular methods applied forthequality design or for life testing of products, in orderto improve products' reliability. This work develops an integrated methodology for improving quality and reliability when the failure time (t) of the product as a response variable is distributed according to a log-epsilon-skew-normal (LESN) distribution. Two factorial design of experiments (2^2) is used to estimate and test the significance of the ANOVA model, based on the likelihood ratio test (LRT). A simulation study and a study with real data are conducted to investigate the performance of the proposed method and the research findings show that the proposed method performs well.*

Keywords: *Design of experiments, reliability, log-epsilon–skew-normal distribution, likelihoodratio test, simulation study.*

JELClassification:C14, C15, C90

1. Introduction

Design of experiments (DOE) can be describedas one of the most common and useful statistical tools for the quality design and life testingof products, and it can also be applied to improve the quality and reliability of products. There are several objectives for reliability experiments, includingimprovementsin reliability and in the robustness of the reliability.Adopting the methodology of DOE for testing reliability can be very helpful in specifying the important factors that directly affect the product itself. Reliability analysis can be a powerful tool indesigning robust products and it is the method we focus on in this paper

As we know, experimental design tools can be used when the failure time (response value) variable at any treatment level is distributed normally. However, Geary (1947)said "normality is a myth, there never was, and never will benormal distribution".In the last few decades there has been a growing interest in finding distributions that can be used as alternatives to the normal distribution, especially in applications thatrequire asymmetric distributions (non-normal distributions). Brown (2001) introduced reliability with failure time distributed according to a skew normal distribution and discussed the reliability properties of the skew normal distribution. Hutson (2004)introduced a new flexible regression model by considering anerror term distributed according to the epsilon skew normal (ESN) distribution. In addition to the estimate of the classical parameters, the skewness parameter has been estimated. Joseph and Yu (2006)consideredthe improvement of reliability with design of experiments using degradation data. They developed an integrated methodology for quality and reliability improvement when the response variable comes from degradation data. Jafari and Hashemi (2011) introduced the so called D-optimal design for when the error term in the simple linear regression follows the skew normal distribution. This design was considered for obtaining an optimal design and estimatingthe parameters of the proposed model. Guo, Niu, Mettasand Ogden (2012) introduced different design types for the design of experiments. Many distributiontypeshavebeenconsideredfor thedescriptionoflifetimes:theWeibull distribution, the lognormal distribution, and the exponential distribution. ReliaSoft Corporation (2015) introduced reliability design of experiments for life tests andthey showed that the failure times of products can follow three types of distribution:theWeibull distribution, the lognormal distribution and the exponential distribution. The design of experiments and thetraditional analysis of variance assumes that the response variable follows the normal distribution. Thus, the dependence on the value of the calculated F in the analysis of variance is used as a base to model the estimatedvalue and determine the important factors.Mohammed et al (2016) studied the factors that have the greatest impact on wheat production in Iraq, by using a factorial experimental design when the random error has a non-normal distribution.

In this research, we propose that the failure time follows the log-epsilon-skew-normal (LESN) distribution, which is an asymmetric probability distribution,and estimate the parameters using the maximum likelihood (MLE) method. Wefirstly explain some concepts in the relationship between the experimental design and reliability. In addition, we use the reliability for an experimental design that contains two factors with two levels, using simulation. We therefore do not rely on the F value to test the model parameters, but we use the likelihood ratio test to determine the important factors that can directly affect the product itself.Inaddition, we calculatethevariance for all parameters in the model, by calculating a Fisher matrix from which we canobtain the variance–covariance matrix.

This paper is organized as follows.In Section 2we present the fundamentals of reliability and the relationship between reliability and designofexperiments, and in Section 3 we briefly introduce the concept of the log-epsilon-skew-normal distribution. In Section 4 we illustrate the LESN with a reliabilityDOE and consider the maximum likelihood estimate when the response variable followsaLESNdistribution. In Section 5 we summarize the results of a simulation study andalso presenta sample data analysis.A brief conclusion is included in Section 6.

2. Reliability and Design of Experiments

As a result of technological progress and intense competition in industry, the study of reliability has been a focus of interest for statisticians, and also for engineers as a result ofits importance in real life. Quality and reliability mean that a product should work in accordance with its production specifications(Baecher& Christian, 2005). Reliability has been developing in recent decades, with a huge contribution fromarmyengineers, especially after World War 2, and this effort has led to more reliable products.

Design of experiments(DOE) has been commonly used in the field of product reliability to identify important factors and to determine the performance of a product or aproduction process(Lamps& Edward, 1993). DOE has been used successfully to improve quality, and can alsobe employed to improve reliability(Condra, 2001). Reliability must be considered along with the design of the product until the finalstages of the production. Companies and the customers must work together to meet adesign specification that leads to a reliable product. For example, temperature is the most important factor in the reliability of tyres of diesel motors, and for such products the experiment depends on one factor. Moststatistically-designed experiments that are used to improve reliability rely on factorial experiments, especially in industrial applications(Hamada, 1995).

Reliability analysis is commonly thought of as an approach formodellingthefailures of existing products(ReliaSoft Corporation, 2015). Using the fitted distribution, failures are mitigated. By adopting the methodology of Design for Reliability (DFR), the analysis can be used as tool to designrobust products that operate with minimal failures. There are two goals: improving reliability (decreasing the mean time to failure) and the ensuring robust reliability (reducing the impact of noise on the reliability variation)(Guo&Mettas, 2007). The important influence factors can be identified experimentally through experimentation,by changing the factor values and observing the resulting reliability. When DOE is used for life testing, the response is the life or failure time (ReliaSoft Corporation, 2008).

3. Log-epsilon-skew-normal distribution

Statistical distributions have a growing importance because oftheirapplied possibilities, and perhaps one of the most important applications is a reliability application that depends on knowledge of the distribution of the data. Distributionsthat are used for reliability are called failure distributions (the exponential, Weibull and gamma distributions). As well as reliability applications with factorial experiments that depend on a knowledge of the distribution of the response variable, there are many studies that integrate the factorial experiment into reliabilityand rely on the distribution of the error, which is the same distributed response variable (dependent variable). In this paper we employ the log-epsilon-skew-normal(LESN) distribution, which is a new family of distributions introduced by Mashtare et al., (2009). LESN traces its roots back to the epsilon-skew–normal distribution(Mudholkar&Hutson, 2000),but the random variable ($\log t$) has the LESN distribution denotedby $\log t \sim LESN\ (\lambda, \sigma, \varepsilon)$.

If there are parameters$\in\ R\ , \sigma > 0$, the probability density function, cumulative density function, and the quintile function of the $LESN$ distribution are given as follows:

$$f_T(t) = \begin{cases} \frac{1}{t\sqrt{2\pi}\,\sigma} exp\left(-\frac{(logt-\lambda)^2}{2\sigma^2(1-\varepsilon)^2}\right) & , \quad if\ 0 < t < e^{\lambda} \\ \frac{1}{t\sqrt{2\pi}\,\sigma} exp\left(-\frac{(logt-\lambda)^2}{2\sigma^2(1+\varepsilon)^2}\right) & , \quad if\ t \geq e^{\lambda} \end{cases} \tag{1}$$

$$aF(t) = \begin{cases} (1-\varepsilon)\Phi\left(\dfrac{logt-\lambda}{\sigma(1-\varepsilon)}\right) & , \quad if\ 0 < t < e^{\lambda} \\ -\varepsilon + (1+\varepsilon)\Phi\left(\dfrac{logt-\theta}{\sigma(1+\varepsilon)}\right) & , \quad if\ t \geq e^{\lambda} \end{cases} \tag{2}$$

And

$$Q_T(u) = exp[\lambda + \sigma Q_0(u)] \tag{3}$$

$$Q_0(u) = F_0^{-1}(u) = \begin{cases} (1-\varepsilon)\Phi^{-1}\left(\dfrac{u}{1-\varepsilon}\right) & if\ 0 < u < (1-\varepsilon)/2 \\ (1+\varepsilon)\Phi^{-1}\left(\dfrac{u+\varepsilon}{1+\varepsilon}\right) & if\ (1-\varepsilon)/2 \leq u < 1 \end{cases} \tag{4}$$

respectively, where$-1 < \varepsilon < 1$.

4. Reliability-DOEanalysis of log-epsilon-skew-normal distributed data

Suppose that the lifetime, t, for a specific product has been found to be log-epsilon-skew-normally distributed. The probability density can be expressed as:

$$f(t) = \begin{cases} \dfrac{1}{t\sqrt{2\pi}\,\sigma} exp\left(-\dfrac{(\log(t)-\lambda)^2}{2\sigma^2(1-\varepsilon)^2}\right) & , \quad if\ 0 < t < e^{\lambda} \\ \dfrac{1}{t\sqrt{2\pi}\,\sigma} exp\left(-\dfrac{(\log(t)-\lambda)^2}{2\sigma^2(1+\varepsilon)^2}\right) & , \quad if\ t \geq e^{\lambda} \end{cases} \tag{5}$$

where λ represents the mean (location parameter) of the log-epsilon-skew-normal of the times-to-failure, σ^2 represents the standard deviation (scale parameter) of the (LESN) distribution of the times-to-failure and ε the skewnessparameter,where-1 <ε<1 (Mashtare et al., 2009).In this paper we study the factorial experiment (2^2) according to the complete randomized design (CRD), which uses the following mathematical model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \tag{6}$$

where$i$ is the number of levels of factor A ($i = 1,2, \dots \dots .. a$),

$j$is the number of levels of factor B ($j = 1,2, \dots \dots .. b$),

Kisthe number of units for each treatment ($k = 1,2, \dots \dots .. n$),

y_{ijk} is the response value for the unit *k* and the levels *I* and *j* of factors A and B, respectively,

$\mu$is the mean effect,

α_i istheleveleffect i of factor A,

β_j istheleveleffect j of factor B,

$(\alpha\beta)_{ij}$isthe effect of the interference level i of factor A and level j of factor B,and

$e_{ijk}$is the experimental error for experimental unit K, which is treated with level i of factor A and level j of factor B.

There are fixed effect models for both factors. Assuming that the response variable represents time-to-failure (T_i), then the following model may be used:

$$T_i = \lambda_i + e_i \tag{7}$$

where $T_i$represents the time-to-failure at the i th treatment level of the factor, λ_i represents the mean value of $\boldsymbol{t_i}$for the ith treatment, ande_i is the random error term.

The model forthe equation shown above is analogous to the ANOVA model $y_i = \lambda_i + e_i$, used in thefactorialexperimental designs for traditional DOE analyses. In this model the random error term, e_i, is not normally distributed because the response$\boldsymbol{T_i}$ is log-epsilon-skew-normally distributed. It is known that the random variable is distributed according to the*LESN*distribution, sothe model can be written as:

$$\log(t_i) = \lambda_i + e_i \tag{8}$$

where $\log(t_i)$ represents the*LESN*time-to-failure at the ith treatment, andλ_i represents the mean (*location parameter*) of the *LESN*of the times-to-failure at the ith treatment.

The random error term, e_i, is non-normally distributed because the response $\log(t_i)$ is non-normally distributed, since the model of the equation given above is identical to the ANOVA model thatis used in traditional DOE analysis and thatcan be applied here as well, just like the R-DOE analysis. According to the two-levelfactorialexperiments, if the factors have only two levels, the notation can be used in the ANOVA model. The R-DOE analysis is based on the distribution assumed for the life data. The purpose of the investigation is to study the effect of the two factors (each at two levels), so the equation can be written as follows:

$$\lambda_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_{12} x_{i1} x_{i2} \tag{9}$$

where$\beta_0$is the intercept term,β_1 is the effect coefficient for the investigated factor,x_1 is the indicator variable representing the first factor,β_2 is the effect coefficient for the second factor,x_2 is the indicator variable representing the second factor, and β_{12} is the interaction effect.

According to this data, an (R-DOE) analysis can be realized for the log-epsilon-skew-normally distributed life data using factorial experiment and maximum likelihood estimation (MLE) techniques.

4.1. Maximumlikelihoodestimate when the response variable follows the LESN distribution

Maximum likelihood methods are generally recommended for calculating parameter estimates for lifetime models. Maximum likelihood methods are statistically optimum for large sample sizes (Meeker& Escobar, 2014),and they easily allow for factorial experiments when the response variable followsanon-normal distribution (Kulkarni&Patil, 2010). In this paperweconsider the factorialexperiment wherethemodel of the lifetime $T_i$is:

$$\log(T_i) = \lambda_i + e_i \tag{10}$$

In this case, λ is the mean value of T_i for the ithtreatment. $e_i$is an error term distributed according to LESN, i.e. $e_i \sim LESN(0, \sigma, \varepsilon)$, so T_i is distributed as the (LESN) defined equation (1), and hence we can write the pdf of the random variable as follows:

$$f_{LESN}(T) = \begin{cases} \frac{1}{T_i\sqrt{2\pi}\,\sigma} exp\left(-\frac{(\log(t_i) - (\boldsymbol{\beta_0}+\boldsymbol{\beta_1 x_{i1}}+\boldsymbol{\beta_2 x_{i2}}))^2}{2\sigma^2(1-\varepsilon)^2}\right) & , \quad if\ 0 < T_i < e^{\lambda} \\ \frac{1}{T_i\sqrt{2\pi}\,\sigma} exp\left(-\frac{(\log(t_i) - (\boldsymbol{\beta_0}+\boldsymbol{\beta_1 x_{i1}}+\boldsymbol{\beta_2 x_{i2}}))^2}{2\sigma^2(1+\varepsilon)^2}\right) & , \quad if\ T_i \geq e^{\lambda} \end{cases} \tag{11}$$

The maximum likelihood estimation method can be used to estimate parameters in R-DOE analyses. The likelihood function is calculated for each observed time to failure t_i, and the parameters of the model are obtained by maximizing the log-likelihood function. The likelihood function for complete data following the log-epsilon-skew-normal distribution is given as:

$$l_{failures} = \prod_{i=1}^{n}(t_i\ , \lambda\ ) \tag{12}$$

$$L_{failur} = \prod_{i=1}^{n}\left[\frac{1}{t_i\sqrt{2\pi}\ \sigma} exp^{\left\{\left(\frac{-(\log(t_i)-(\beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}))^2}{2\ \sigma^2(1-\varepsilon)^2}\right) I_{\left(0<t<e^{\lambda}\right)} + \left(\frac{-(\log(t_i)-(\beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}))^2}{2\ \sigma^2(1+\varepsilon)^2}\right) I_{\left(t \geq e^{\lambda}\right)}\right\}}\right] \tag{13}$$

where $\lambda = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}$, (14)
n is the total number of observed times-to-failure,
$t_i$is the time of the ith failure, and
λ is the life characteristic.

In this paper we use the factorial experiment (2^2) with the exclusion of the interaction between factors.

$$\prod_{i=1}^{4}\left[\frac{1}{t_i\sqrt{2\pi}\ \sigma}e^{-\frac{1}{2}\frac{(\log(t_i)-(\beta_0+\beta_1x_{i1}+\beta_2x_{i2}))^2}{\sigma^2(1+\varepsilon)^2}}\right]_{I_{(t\geq e^{\lambda})}} + \prod_{i=1}^{4}\left[\frac{1}{t_i\sqrt{2\pi}\ \sigma}e^{-\frac{1}{2}\frac{(\log(t_i)-(\beta_0+\beta_1x_{i1}+\beta_2x_{i2}))^2}{\sigma^2(1-\varepsilon)^2}}\right]_{I_{(0<t\leq e^{\lambda})}} \quad (15)$$

Then the log-likelihood function is $\partial(\beta_0\ ,\beta_1,\beta_2,\sigma,\varepsilon)$,

$$L=\sum_{i=1}^{4}ln\left[\frac{1}{t_i\sqrt{2\pi}\ \sigma}e^{-\frac{1}{2}\frac{(log(t_i)-(\beta_0+\beta_1x_{i1}+\beta_2x_{i2}))^2}{\sigma^2(1+\varepsilon)^2}}\right]_{I_{(t\geq e^{\lambda})}} + \sum_{i=1}^{4}ln\left[\frac{1}{t_i\sqrt{2\pi}\ \sigma}e^{-\frac{1}{2}\frac{(log(t_i)-(\beta_0+\beta_1x_{i1}+\beta_2x_{i2}))^2}{\sigma^2(1-\varepsilon)^2}}\right]_{I_{\left(0<t\leq e^{\lambda}\right)}} \quad (16)$$

To obtain the MLE estimate of the parameters $(\beta_0\ ,\beta_1,\beta_2,\sigma,\varepsilon)$, the log-likelihood function must be differentiated with respect to the parameters. Because of difficulties in obtaining the coefficients$\beta_0\ ,\beta_1,\beta_2$, we can use numerical methods.The coefficients $\beta_0\ ,\beta_1,\beta_2$can be obtained by using the numerical methods and the methodology of MudholkarandHutson (2000), and the MLE_s can be used for the parameters. We can then find an estimate of σ,ε by using the order statistic, as follows:

$$\hat{\varepsilon}=\frac{\left[\sum_{i=j+1}^{n}(\log(t_i)-\lambda_i)^2\right]^{1/3}-\left[\sum_{i=1}^{j}(log(t_i)-\lambda_i)^2\right]^{1/3}}{\left[\sum_{i=j+1}^{n}(log(t_i)-\lambda_i)^2\right]^{1/3}+\left[\sum_{i=1}^{j}(log(t_i)-\lambda_i)^2\right]^{1/3}} \quad (17)$$

$$\hat{\sigma}=\frac{1}{4n}\left\{\left[\sum_{i=1}^{j}(\log(t_i)-\lambda_i)^2\right]^{1/3}+\left[\sum_{i=j+1}^{n}(log(t_i)-\lambda_i)^2\right]^{1/3}\right\}^3 \quad (18)$$

where $\lambda=\beta_0+\beta_1x_{i1}+\beta_2x_{i2}$.

Unfortunately, we note that using the maximum likelihood method directly makes it difficult to obtain acloseestimationfrom the expression for the parameters of the model (β_i), so we must adopt a numerical method such as the Nelder–Mead optimization method (Nelder& Mead, 1965). Toperformthatestimation we use R programming to find the MLEfor the parameters of interest.

Maximizing (16) numerically requires initial values to be set for $\beta_0\ ,\beta_1,\beta_2,\varepsilon$ and σ. In addition, the initial values for the skewness and scale parameters are $-1<\varepsilon<1$ and $\sigma>0$.

4.2. Fisher matrix and statistical testing

Fisher information is widely used in optimal experimental design. Because of the reciprocity of the estimator-variance and the Fisher information, minimizing the variance corresponds to maximizing the information. In this work weusetheFisher matrix to get the variance. In order to calculate the confidence intervals, if $\hat{\lambda}$is the MLE estimate of any parameter λ, then the $(1-\alpha)100\%$ two-sided confidence intervals on the parameter are:

$$\hat{\lambda} - z_{\alpha/2}.\sqrt{var(\hat{\lambda})} < \lambda < \hat{\lambda} + z_{\alpha/2}.\sqrt{var(\hat{\lambda})} \tag{19}$$

where $var(\hat{\lambda})$ represents the variance of $\hat{\lambda}$ and $z_{\alpha/2}.\sqrt{var(\hat{\lambda})}$ is the critical value corresponding to a significance level of $\alpha/2$ on the standard normal distribution. Following Meeker and Escobar (1998), the Fisher information matrix is obtained from the log-likelihood function as follows:

$$F = \begin{bmatrix} -\frac{\partial^2 L}{\partial \beta_0^2} & -\frac{\partial^2 L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 L}{\partial \beta_0 \partial \beta_2} & -\frac{\partial^2 L}{\partial \beta_0 \partial \sigma} & -\frac{\partial^2 L}{\partial \beta_0 \partial \varepsilon} \\ & -\frac{\partial^2 L}{\partial \beta_1^2} & -\frac{\partial^2 L}{\partial \beta_1 \partial \beta_2} & -\frac{\partial^2 L}{\partial \beta_1 \partial \sigma} & -\frac{\partial^2 L}{\partial \beta_1 \partial \varepsilon} \\ & & -\frac{\partial^2 L}{\partial \beta_2^2} & -\frac{\partial^2 L}{\partial \beta_2 \partial \sigma} & -\frac{\partial^2 L}{\partial \beta_2 \partial \varepsilon} \\ & & & -\frac{\partial^2 L}{\partial\, \sigma^2} & -\frac{\partial^2 L}{\partial \sigma \partial \varepsilon} \\ & & & & -\frac{\partial^2 L}{\partial\, \varepsilon^2} \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{4(1+\varepsilon)^2}{\sigma^2(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{\sum_{i=1}^4 x_{i1}(1+\varepsilon)^2}{\sigma^2(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{\sum_{i=1}^4 x_{i2}(1+\varepsilon)^2}{\sigma^2(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{2[\sum_{i=1}^4 (W)](1+\varepsilon)^2}{\sigma^3(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{\frac{-4}{\sigma^2}[\sum_{i=1}^4 (W)^2](1+3\varepsilon^2)}{(1+\varepsilon)^3(1-\varepsilon)^3} \\ & \frac{\sum_{i=1}^4 x_{i1}^2(1+\varepsilon)^2}{\sigma^2(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{\sum_{i=1}^4 x_{i1}x_{i2}(1+\varepsilon)^2}{\sigma^2(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{2[\sum_{i=1}^4 x_{i1}(W)](1+\varepsilon)^2}{\sigma^3(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{\frac{-4}{\sigma^2}[\sum_{i=1}^4 x_{i1}(W)^2](1+3\varepsilon^2)}{(1+\varepsilon)^3(1-\varepsilon)^3} \\ & & \frac{\sum_{i=1}^4 x_{i2}^2(1+\varepsilon)^2}{\sigma^2(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{2[\sum_{i=1}^4 x_{i2}(W)](1+\varepsilon)^2}{\sigma^3(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{\frac{-4}{\sigma^2}[\sum_{i=1}^4 x_{i2}(W)^2](1+3\varepsilon^2)}{(1+\varepsilon)^3(1-\varepsilon)^3} \\ & & & \frac{8}{\sigma^2} + \frac{(-3)[\sum_{i=1}^4 (W)^2](1+\varepsilon)^2}{\sigma^4(1+\varepsilon)^2(1-\varepsilon)^2} & \frac{-2\frac{1}{\sigma^3}[\sum_{i=1}^4 (W)^2](1+3\varepsilon^2)}{(1+\varepsilon)^3(1-\varepsilon)^3} \\ & & & & \frac{-\frac{1}{\sigma^2}[\sum_{i=1}^4 (W)^2](1+6\varepsilon^2+\varepsilon^4)}{3(1-4\varepsilon^2-2\varepsilon^6+\varepsilon^8)} \end{bmatrix}$$

where $W = (ln(t_i) - (\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2})$.

The classical ANOVA test is widely used in the design of experiments, and the F-statistics are calculated by assuming a normal distribution model for the errors. The significance of the ANOVA test depends on the value of the F-statistic. In real life, many of the characteristics of products(like the failure time) do not follow anormal distribution for the error model. We will therefore not rely on the calculated value ofF to findthe significance of the model, butwe will use the likelihood ratio test. Likelihood ratio tests are widely applicable tests related to maximum likelihood estimation. While likelihood ratio test procedures are very useful and widely applicable, the computations are difficult to perform by hand, especially for failuredata. The likelihood ratio (LR) is defined as the ratio of the likelihood under the null hypothesis to the likelihood under the alternative hypothesis (Freeman, 2010),and the likelihood ratio testcan be used to test the significance of each effect. The LR statistics are given by:

Likelihood ratio test (LRT) $=-2ln\frac{L_{(reduced\ model)}}{L_{(Full\ model)}}$, LRT$= -2\ ln\frac{L(\hat{\lambda}_{(-i)})}{L(\hat{\lambda})}$

where$\lambda^{\wedge}$is the vector for all parameters,$\hat{\lambda}_{(-i)}$is the vector of all parameter estimates excluding λ_i,

$L(\lambda^{\wedge})$is the value of the likelihood function when all parameters are included in the model, and

$L_{(\lambda^{\wedge}_{-i})}$is the value of the likelihood function when all parameters except λ_i are included in the model.

This follows the chi-square (χ^2)distribution with k degrees of freedom,and the hypothesis test is based on the following:

$H_0: \lambda_i = 0$ vs. $H_1: \lambda_i \neq 0$.

If the null hypotheses,$H_{0,}$, is true, then the ratio $-2\ln L(\hat{\lambda}_{(-i)})/L(\hat{\lambda})$, follows the chi-squared distribution with degree of freedom,and the null hypothesis is rejected if$LR > \chi^2_{d-1,\alpha}$.

5. Application

5.1 Simulation study

In order to illustrate the maximum likelihood estimation of the parameters $(\beta_0, \beta_1, \beta_2, \sigma, \varepsilon)$ of the log-epsilon-skew-normal distribution for reliability, we conduct a simulationstudyusingthe R software.The goal of this simulation is to study the reliability of factorial experiment (2^2),and to estimatetheparameters when the response variable (time-to-failure) follows the LESN distribution. At first we assume that $T \sim LESN$, and repeat each experiment (IT=1000)for all the simulation experiments. Then we identify some of the default values for the parameters that we need in this distribution, as well as changing the number of repetitions for each experiment(r=1,2,4,10)and the default values for the skew parameter ($\varepsilon = 0.2,\ 0.5,\ 0.8$). When giving default values for the factors (A,B) we represent the parameters (β_1, β_2). After estimating the parameters, we findtheFisher matrix and calculate the confidence intervals (upper and lower) in order to get the parameters estimationand perform the likelihoodratio (LR) test. We observe the changes in the number of repetitions for each experiment (r= 1, 2, 4, 10), give default values forthe skew parameter ($\varepsilon = -0.2,\ -0.5, -0.8$), and also define default values forthe factors (A,B), represented by the parameters (β_1, β_2). After estimating the parameters, finding the Fisher matrix and calculating the confidence intervals (upper and lower),we can estimate the parameters and perform the likelihood ratio(LR) test.

Tables 1 and 2 summarize the MLE for the parameters (β_0 ,β_1, β_2, ε and σ), and wecanseethat the parameter estimates are close to the default values in the designed algorithm for this paper.We note that there is a simple difference, and sometimes an increase in variance (σ^2) when we increase the number of repetitions (r). Figure 1 in the Annex graphically illustrate the (LRT) values for the factors (A,B), andsupporttheresults obtained in terms of stability and convergence to the simulation experiments.

Table 1: Estimates of Expected Values and Confidence Intervals when $\varepsilon > 0$

Repetitions	ε	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}$	$\hat{\varepsilon}$
1	0.2	0.8251005	1.28286	1.009132	10.0549085	0.184804
	Lower	0.238701	1.0583	0.835104	9.161617	0.150824
	Upper	1.4115	1.50742	1.18316	10.9482	0.218817
	0.5	0.379263	0.797381	0.5349445	4.81363475	0.370594
	Lower	0.21816	0.642681	0.331662	2.5137715	0.264197
	Upper	0.540366	0.952081	0.738227	7.113498	0.476948
	0.8	0.1671825	1.0839119	0.3426421	11.3116224	0.787254
	Lower	0.010884	0.2100228	0.2715242	9.8704548	0.670160
	Upper	0.323481	1.957801	0.41376	12.75279	0.972640
2	0.2	0.3835477	0.6086794	0.5504023	11.46434714	0.3313935
	Lower	0.2835777	0.5009808	0.3391643	9.88170427	0.1698135
	Upper	0.4835177	0.716378	0.7616403	13.04699	0.4963010
	0.5	0.4247641	0.70579194	0.620659855	13.59069935	0.4784541
	Lower	0.1077341	0.49987628	0.47245431	10.4165107	0.2350645
	Upper	0.7417941	0.9117076	0.7688654	16.764888	0.7220542

	0.8	0.419759295	1.502663955	0.307808035	15.59209743	0.6474366
	Lower	0.18790429	1.033829009	0.50043277	12.01818885	0.02774647
	Upper	0.6516143	1.9714989	0.1151833	19.166006	1.2661917
4	0.2	0.2626056	0.5724389	0.536236505	13.91578725	0.2343062
	Lower	0.1561992	0.35537982	0.20519361	7.6311875	0.11454061
	Upper	0.369012	0.7894981	0.8672794	20.200387	0.35420134
	0.5	0.346707115	0.37682807	0.74361335	15.7842955	0.3553636
	Lower	0.13102203	0.62918529	0.0916161	8.336712691	0.28270771
	Upper	0.5623922	0.12447085	1.3956106	23.2318783	0.4483012
	0.8	-0.48837265	0.87897705	1.203681565	17.38765071	0.6461795
	Lower	-0.671832	0.3905401	0.51193013	9.30285141	0.3361085
	Upper	-0.3049133	1.367414	1.895433	25.47245	0.9573231
10	0.2	0.466957045	0.6072443	0.547520025	9.3268418	0.2636345
	Lower	0.12172489	0.260097	0.18856005	7.5874597	0.1980783
	Upper	0.8121892	0.954389	0.90648	11.0662239	0.3279419
	0.5	0.36952925	0.5798418	0.93004085	17.3596318	0.4611297
	Lower	0.1900185	0.32407693	0.2810888	9.3616526	0.3082499
	Upper	0.54904	0.83560681	1.5789929	25.357611	0.6114148
	0.8	0.57175078	0..4426535	0.5663246	16.896615	0.7441453
	Lower	0.49289228	0.2630461	0.3835246	15.664261	0.6508885
	Upper	0.65060928	0.622261	0.7491246	18.128969	0.8373494

Table 2: Estimates of Expected Values and Confidence Intervalswhenε < 0

Repetitions	ε	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}$	$\hat{\varepsilon}$
1	-0.2	0.758924305	1.16736	0.2722282	10.30430385	-0.266419
	Lower	0.39602261	0.822764	0.1292513	9.4811037	-0.415878
	Upper	1.121826	1.511964	0.4152051	11.127504	-0.117431
	-0.5	0.47229325	1.087640595	1.08184355	14.78394643	-0.4565695
	Lower	0.2068968	0.9309893	0.6497155	11.81865118	-0.5303414
	Upper	0.7376897	1.24429189	1.5139716	17.74924167	-0.3827089
	-0.8	1.406651625	0.3607313	1.78853921	13.80661576	-0.68673419
	Lower	0.8975185	0.2565321	0.7396435	8.83344711	-0.9680719
	Upper	1.91578475	0.4649316	2.83743492	18.77978441	-0.4053466
2	-0.2	-0.5868982	1.360552395	0.81218298	21.26857784	-0.2392330
	Lower	-0.8322083	0.77019489	0.52224546	17.41693118	-0.32433783
	Upper	-0.3415881	1.9509099	1.1021205	25.1202245	-0.1543103
	-0.5	0.974178075	1.3512590	1.174655817	22.47395904	-0.43638441
	Lower	0.42374805	0.89797096	0.748415484	19.66353418	-0.7524244
	Upper	1.5246081	1.80449971	1.60089615	25.2843839	-0.1205719
	-0.8	0.89121184	1.588399545	1.604865137	15.30757826	-0.7532408
	Lower	0.23585058	0.87116692	0.783717524	12.92920651	-0.9464369
	Upper	1.5465731	2.30563217	2.42601275	17.68595	-0.5613775
4	-0.2	0.22593232	0.602125	0.568417	15.415057	-0.2461046
	Lower	0.01738032	0.235525	0.1724145	11.584914	-0.3032236
	Upper	0.43448432	0.968725	0.9644145	19.2452	-0.1891558
	-0.5	0.5096188	1.68013518	0.6702267	15.0933666	-0.4603496

	Lower	0.2924388	0.96085124	0.355412	11.9976662	-0.730028
	Upper	0.7267988	2.39941912	0.9850412	18.189067	-0.1910657
	-0.8	0.71580255	0.02480171	1.5463963	13.51744315	-0.7903356
	Lower	0.467641	0.18183908	0.9684963	11.3283073	-0.9327625
	Upper	0.9639641	0.31419008	2.1242963	15.706579	-0.6482081
10	-0.2	0.583506655	0.608486130	0.603123677	13.32776587	-0.2751807
	Lower	0.24488483	0.38767031	0.268258853	10.07742893	-0.3654827
	Upper	0.92212848	0.8293031	0.9379885	16.5781028	-0.1863808
	-0.5	0.622506825	0.80246715	1.129484115	20.04289449	-0.475435
	Lower	0.3268848	0.22566755	0.65028413	12.75462978	-0.8049397
	Upper	0.91812885	1.37926675	1.6086841	27.3311592	-0.1468125
	-0.8	0.8107546	1.8705999	2.1360576	14.7090549	-0.7157391
	Lower	0.4613546	0.7825999	0.6174576	11.0690574	-0.936296
	Upper	1.1601546	2.9585999	3.6546576	18.3490524	-0.4937762

5.2 Data illustration

To illustrate the use of reliability with factorial experiments we collected data from a factorial experiment(2^2) that wasconducted in the Al-Diwaniyahtyre factory in Iraq for the years 2015-2016 (60 observations). For this study, the most important factorswerethoseaffecting the life of a tyre (t). Two factors were collected at two levels: a speed factor (A) with two levels, respectively a high level (1) and a low level (-1), and pressure factor (B) with two levels, respectively highlevel(1) and a low level (-1). The speed changed every ten minutes over an hour while the air pressure was changedin the tyre. In this way the life of a tyre (the time until it burst at somespeedand pressure) was recorded. The following Table 3shows the results that were obtained, which represent the mean for each level and each of the factors.

Table3:Means for each of the factors levels

Standard order	Factor (A)	Factor (B)	Time to failure
1	-1	-1	36.75
2	1	-1	40.18
3	-1	1	36.666
4	1	1	41.25

Since the lifetime (the tyre life) follows the LESN distribution, the probability density function for this distribution is expressedaccording to equation (1) , where λrepresents the mean of the log-epsilon-skew-normal of the times-to-failure.

$\lambda = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}$,$\beta_1$is the effect coefficient for factor A (speed), and β_2 is the effect coefficient for factor B (pressure). With the interaction between A and

Bbeingexcluded,weuse the MLE method mentioned in Section 4for the estimation of the parameters ($\beta_0, \beta_1, \beta_2, \sigma, \varepsilon$). We also calculate the Fisherinformation matrix, and the variance–covariance matrix that can be obtained by taking the inverse of the Fisher matrix F, through which we get thevariance. In this way we get the confidence bounds (upper and lower), and use R programming to analyse our data. Table 4 shows the results obtained.

Table 4: MLE Coefficientforthelife of a tyre

Parameter	MLE Coeff.	LowerCI	UpperCI	Likelihood Ratio	P value
$\hat{\beta}_0$	3.5524	3.5229733	3.5818266	-	-
$\hat{\beta}_1$(A)	0.0486	0.0191733	0.0780266	7.30500	0.068
$\hat{\beta}_2(B)$	0.15410	0.1246733	0.183526	73.292662	0.0000
$\hat{\sigma}$	0.032600	0.0198407	0.045320	-	-
$\hat{\varepsilon}$	0.0437327	0.028777	0.059159	-	-

It is clear that the estimated values of the parameters of the model ($\lambda = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i1}$), representing the mean times to failure,$\hat{\beta}_0$ =3.5524, $\hat{\beta}_1$ = 0.0486,$\hat{\beta}_2$ = 0.1541, $\hat{\sigma} = 0.03600$ and $\hat{\varepsilon} = 0.0437327$. In the likelihood ratio test, the main factor (LR_A) represents the speed.LR_A= 7.30500 with a p-value of 0.06, which is not significant at the 5% level.The likelihood ratio of the second main factor(LR_B) represents the pressure. $LR_B = 73.2926$ with p-value of 0.0000, which is significant at the 5% level.

6. Conclusions

Design of experimentsis widely applied for analysing the most relevant factors affecting products' performance, quality or reliability. It is widely recognizedthat non-normal distributions, particularly asymmetric ones, occur very frequently in practice. Wehaveexplainedsome of the concepts used in factorial experiments, and the relationship between design of experiments and reliability. In addition, we have developed some designs with response variables that follow a log-epsilon-skew-normal distribution and describe the time-to-failure model to develop an integrated method that, in turn, will be very helpful in improving both the quality and the reliability of products.Our assumption was that the failure time for the product is a response variable in the factorial experiments. MLE is an important method for estimating parameters in DOE. This is therefore emphasized. For DOE involving censored data, we also noted that the likelihood ratio test(LRT),from which we can determine the important factors, is the best method when the failure timeis non-normally distributed.

We conducted a case study on some factors that affected the lifetime of a tyre. We concluded that the pressure factor value (0.15410) had a direct impactonthe life of a tyre at the 5% level of significance, with confidence bounds (0.183526, 0.1246733).This research is seminal because it brings together reliability analysis and the statistical design of experiments forwhen the response variable follows an LESN distribution. An area for future research is to applythisto some skewdistributions as well as to other factorial experiments.

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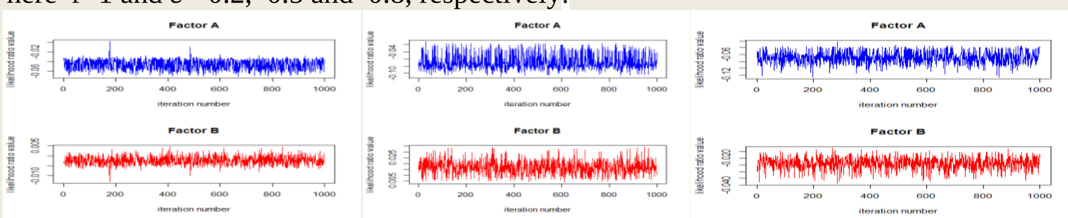
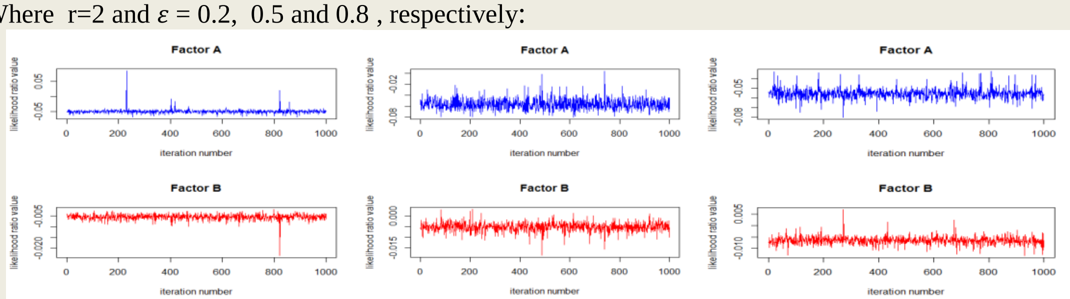
Annex

Figure 1: Likelihood ratio to factor (A and B),when $\varepsilon > 0$.

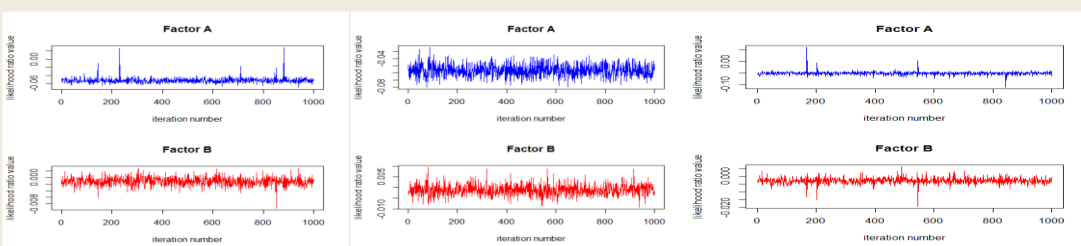
Where r=1 and ε= 0.2, 0.5 and 0.8, respectively:



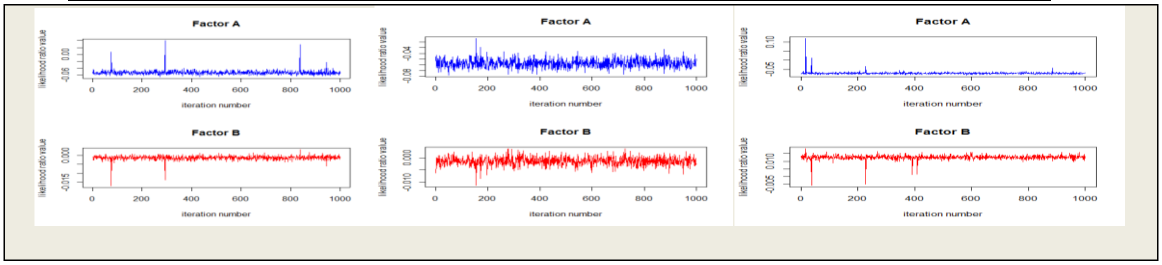
Where r=2 and ε = 0.2, 0.5 and 0.8 , respectively:



Where r =4 and ε = 0.2, 0.5 and 0.8 , respectively:



Where r =10 andε = 0.2, 0.5 and 0.8, respectively:


Factor A
Factor A
Factor A
Factor B
Factor B
Factor B
likelihood ratio value
iteration number
0 200 400 600 800 1000